

PICARD GROUPS OF CERTAIN STABLY PROJECTIONLESS C*-ALGEBRAS

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ABSTRACT. We compute Picard groups of several nuclear and non-nuclear simple stably projectionless C*-algebras. In particular, the Picard group of Razak-Jacelon algebra \mathcal{W}_2 is isomorphic to a semidirect product of $\text{Out}(\mathcal{W}_2)$ with \mathbb{R}_+^\times . Moreover, for any separable simple nuclear stably projectionless C*-algebra with a finite dimensional lattice of densely defined lower semicontinuous traces, we show that \mathcal{Z} -stability and strict comparison are equivalent. (This is essentially based on the result of Matui and Sato, and Kirchberg's central sequence algebras.) This shows if A is a separable simple nuclear stably projectionless C*-algebra with a unique tracial state (and no unbounded trace) and has strict comparison, the following sequence is exact:

$$1 \longrightarrow \text{Out}(A) \longrightarrow \text{Pic}(A) \longrightarrow \mathcal{F}(A) \longrightarrow 1$$

where $\mathcal{F}(A)$ is the fundamental group of A .

1. INTRODUCTION

Let A be a C*-algebra. Brown, Green and Rieffel introduced the Picard group $\text{Pic}(A)$ of A in [5]. We say that an automorphism α of A is *inner* if there exists a unitary element u in the multiplier algebra $M(A)$ of A such that $\alpha(a) = uau^*$ for any $a \in A$. Let $\text{Inn}(A)$ denote the set of inner automorphisms of A , and let $\text{Out}(A) = \text{Aut}(A)/\text{Inn}(A)$. They showed that if A is σ -unital, then $\text{Pic}(A)$ is isomorphic to $\text{Out}(A \otimes \mathbb{K})$. Kodaka computed Picard groups of several unital C*-algebras in [20], [21] and [22]. In particular he computed the Picard groups of the irrational rotation algebras A_θ . If θ is not quadratic irrational number, then $\text{Pic}(A)$ is isomorphic to $\text{Out}(A_\theta)$ and if θ is a quadratic number, then $\text{Pic}(A_\theta)$ is isomorphic to $\text{Out}(A_\theta) \rtimes \mathbb{Z}$. Kodaka considered the following set

$$\text{FP}/\sim = \{[p] \mid p \text{ is a full projection in } A \otimes \mathbb{K} \text{ such that } p(A \otimes \mathbb{K})p \cong A\}$$

where $[p]$ is the Murray-von Neumann equivalence class of p and showed that if $\text{Out}(A)$ is a normal subgroup of $\text{Out}(A \otimes \mathbb{K})$, then FP/\sim has a suitable group structure and the following sequence is exact:

$$1 \longrightarrow \text{Out}(A) \longrightarrow \text{Pic}(A) \longrightarrow \text{FP}/\sim \longrightarrow 1.$$

Note that there exists a simple unital AF algebra A with a unique tracial state such that FP/\sim of A does not have any suitable group structure. K-theoretical method enables us to show that $\text{Out}(A)$ is a normal subgroup of $\text{Out}(A \otimes \mathbb{K})$ (see [20, Proposition 1.5]).

The set of FP/\sim is similar to the fundamental group $\mathcal{F}(M)$ of a II_1 factor M introduced by Murray and von Neumann in [26]. Watatani and the author introduced the fundamental group $\mathcal{F}(A)$ of a simple unital C*-algebra A with a unique tracial state τ based on Kodaka's results. The fundamental group $\mathcal{F}(A)$

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is defined as the set of the numbers $\tau \otimes \text{Tr}(p)$ for some projection $p \in M_n(A)$ such that $pM_n(A)p$ is isomorphic to A . We showed that $\mathcal{F}(A)$ is a multiplicative subgroup of \mathbb{R}_+^\times and computed fundamental groups of several C^* -algebras in [29]. Moreover we showed that any countable subgroup of \mathbb{R}_+^\times can be realized as the fundamental group of a separable simple unital C^* -algebra with a unique trace in [30]. Note that the fundamental groups of separable simple unital C^* -algebras are countable. Furthermore the author introduced the fundamental group of a simple stably projectionless C^* -algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ in [27]. If τ is normalized and A is σ -unital, then the fundamental group of $\mathcal{F}(A)$ of A is defined as the set of the numbers $d_\tau(h)$ for some positive element $h \in A \otimes \mathbb{K}$ such that $\overline{h(A \otimes \mathbb{K})h}$ is isomorphic to A where d_τ is the dimension function defined by τ . Note that if A is unital, then this definition coincides with the previous definition and there exist separable simple stably projectionless C^* -algebras such that their fundamental groups are equal to \mathbb{R}_+^\times . The fundamental group of a II_1 factor M is equal to the set of trace-scaling constants for automorphisms of a II_∞ factor $M \otimes B(\mathcal{H})$. This characterization shows that the fundamental groups of II_1 factors are related to the structure theorem for type III_λ factors where $0 < \lambda \leq 1$ (see [41] and [42]). We have a similar characterization, that is, if A is σ -unital, then the fundamental group of A is equal to the set of trace scaling constants for automorphisms of $A \otimes \mathbb{K}$.

In this paper we shall compute Picard groups of several nuclear and non-nuclear simple stably projectionless C^* -algebras. In the case of stably projectionless C^* -algebras, the theory of the Cuntz semigroup enables us to compute Picard groups of several examples. We shall show that if A is a separable simple exact \mathcal{Z} -stable stably projectionless C^* -algebra with a unique tracial state τ and no unbounded trace, then the following sequence is exact:

$$1 \longrightarrow \text{Out}(A) \longrightarrow \text{Pic}(A) \longrightarrow \mathcal{F}(A) \longrightarrow 1.$$

Since there exists a unital simple \mathcal{Z} -stable algebra A with a unique tracial state such that $\text{Out}(A)$ is not a normal subgroup of $\text{Pic}(A)$, \mathcal{Z} -stable stably projectionless C^* -algebras are more well-behaved than unital stably finite \mathcal{Z} -stable C^* -algebras. Let \mathcal{W}_2 be the Razak-Jacelon algebra studied in [12], [35] and [36], which has trivial K -groups and a unique tracial state and no unbounded trace. Then \mathcal{W}_2 is \mathcal{Z} -stable, and hence the sequence above is exact in this case. Moreover we shall show that the exact sequence above splits. Therefore $\text{Pic}(\mathcal{W}_2)$ is isomorphic to $\text{Out}(\mathcal{W}_2) \rtimes \mathbb{R}_+^\times$.

Based on the result of Matui and Sato, and Kirchberg's central sequence algebras, for any separable simple infinite-dimensional non-type I nuclear C^* -algebra with a finite dimensional lattice of densely defined lower semicontinuous traces, we shall show that \mathcal{Z} -stability and strict comparison are equivalent. (It is important to consider property (SI).) In particular, if A is a simple C^* -algebra with a finite dimensional lattice of densely defined lower semicontinuous traces in the class of Robert's classification theorem ([35, Corollary 6.2.4]), then A is \mathcal{Z} -stable. Moreover we see that there are many examples that the sequence above are exact. But we do not know whether the exact sequence above splits in this case. This question is related to the existence of a one parameter trace scaling automorphism group of $A \otimes \mathbb{K}$. In the final part of this paper we shall give some remarks and a reason of the notation of \mathcal{W}_2 . Some results show every separable simple \mathcal{Z} -stable stably projectionless C^* -algebra A with a unique tracial state has similar properties of (McDuff) II_1 factors.

2. THE PICARD GROUP

In this section we shall review basic facts on the Picard groups of C*-algebras introduced by Brown, Green and Rieffel in [5] and some results in [27].

Let A be a C*-algebra and \mathcal{X} a right Hilbert A -module. For $\xi, \eta \in \mathcal{X}$, a "rank one operator" $\Theta_{\xi, \eta}$ is defined by $\Theta_{\xi, \eta}(\zeta) = \xi \langle \eta, \zeta \rangle_A$ for $\zeta \in \mathcal{X}$. We denote by $K_A(\mathcal{X})$ the closure of the linear span of "rank one operators" $\Theta_{\xi, \eta}$ and by \mathbb{K} the C*-algebra of compact operators on an infinite-dimensional separable Hilbert space. Let H_A denote the standard Hilbert module $\{(x_n)_{n \in \mathbb{N}} \mid x_n \in A, \sum x_n^* x_n \text{ converges in } A\}$ with an A -valued inner product $\langle (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \rangle = \sum x_n^* y_n$. Then there exists a natural isomorphism of $A \otimes \mathbb{K}$ to $K_A(H_A)$.

Let A and B be C*-algebras. An A - B -equivalence bimodule is an A - B -bimodule \mathcal{F} which is simultaneously a full left Hilbert A -module under a left A -valued inner product ${}_A \langle \cdot, \cdot \rangle$ and a full right Hilbert B -module under a right B -valued inner product $\langle \cdot, \cdot \rangle_B$, satisfying ${}_A \langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B$ for any $\xi, \eta, \zeta \in \mathcal{F}$. We say that A is *Morita equivalent* to B if there exists an A - B -equivalence bimodule. It is easy to see that $K_B(\mathcal{F})$ is isomorphic to A . A dual module \mathcal{F}^* of an A - B -equivalence bimodule \mathcal{F} is a set $\{\xi^*; \xi \in \mathcal{F}\}$ with the operations such that $\xi^* + \eta^* = (\xi + \eta)^*$, $\lambda \xi^* = (\overline{\lambda} \xi)^*$, $b \xi^* a = (a^* \xi b^*)^*$, ${}_B \langle \xi^*, \eta^* \rangle = \langle \eta, \xi \rangle_B$ and $\langle \xi^*, \eta^* \rangle_A = {}_A \langle \eta, \xi \rangle$. The bimodule \mathcal{F}^* is a B - A -equivalence bimodule. We refer the reader to [33] and [34] for the basic facts on equivalence bimodules and Morita equivalence. For A - A -equivalence bimodules \mathcal{E}_1 and \mathcal{E}_2 , we say that \mathcal{E}_1 is isomorphic to \mathcal{E}_2 as an equivalence bimodule if there exists a \mathbb{C} -linear one-to-one map Φ of \mathcal{E}_1 onto \mathcal{E}_2 with the properties such that $\Phi(a \xi b) = a \Phi(\xi) b$, ${}_A \langle \Phi(\xi), \Phi(\eta) \rangle = {}_A \langle \xi, \eta \rangle$ and $\langle \Phi(\xi), \Phi(\eta) \rangle_A = \langle \xi, \eta \rangle_A$ for $a, b \in A$, $\xi, \eta \in \mathcal{E}_1$. The set of isomorphic classes $[\mathcal{E}]$ of the A - A -equivalence bimodules \mathcal{E} forms a group under the product defined by $[\mathcal{E}_1][\mathcal{E}_2] = [\mathcal{E}_1 \otimes_A \mathcal{E}_2]$. We call it the *Picard group* of A and denote it by $\text{Pic}(A)$. The identity of $\text{Pic}(A)$ is given by the A - A -bimodule $\mathcal{E} := A$ with ${}_A \langle a_1, a_2 \rangle = a_1 a_2^*$ and $\langle a_1, a_2 \rangle_A = a_1^* a_2$ for $a_1, a_2 \in A$. The inverse element of $[\mathcal{E}]$ in the Picard group of A is the dual module $[\mathcal{E}^*]$. Let α be an automorphism of A , and let $\mathcal{E}_\alpha^A = A$ with the obvious left A -action and the obvious A -valued inner product. We define the right A -action on \mathcal{E}_α^A by $\xi \cdot a = \xi \alpha(a)$ for any $\xi \in \mathcal{E}_\alpha^A$ and $a \in A$, and the right A -valued inner product by $\langle \xi, \eta \rangle_A = \alpha^{-1}(\xi^* \eta)$ for any $\xi, \eta \in \mathcal{E}_\alpha^A$. Then \mathcal{E}_α^A is an A - A -equivalence bimodule. For $\alpha, \beta \in \text{Aut}(A)$, \mathcal{E}_α^A is isomorphic to \mathcal{E}_β^A if and only if there exists a unitary $u \in M(A)$ such that $\alpha = \text{ad } u \circ \beta$. Moreover, $\mathcal{E}_\alpha^A \otimes \mathcal{E}_\beta^A$ is isomorphic to $\mathcal{E}_{\alpha \circ \beta}^A$. Hence we obtain an homomorphism ρ_A of $\text{Out}(A)$ to $\text{Pic}(A)$. An A - B -equivalence bimodule \mathcal{F} induces an isomorphism Ψ of $\text{Pic}(A)$ to $\text{Pic}(B)$ by $\Psi([\mathcal{E}]) = [\mathcal{F}^* \otimes \mathcal{E} \otimes \mathcal{F}]$ for $[\mathcal{E}] \in \text{Pic}(A)$. Therefore if A is Morita equivalent to B , then $\text{Pic}(A)$ is isomorphic to $\text{Pic}(B)$. Brown, Green and Rieffel showed that if A is σ -unital, then $\text{Pic}(A)$ is isomorphic to $\text{Out}(A \otimes \mathbb{K})$ (see [5, Theorem 3.4 and Corollary 3.5]).

If A is σ -unital, then for any A - A -equivalence bimodule \mathcal{E} there exists a positive element h in $A \otimes \mathbb{K}$ such that \mathcal{E} is isomorphic to $\overline{h H_A}$ as a right Hilbert A -module. Note that $\overline{h(A \otimes \mathbb{K})h}$ is isomorphic to A and $\overline{h H_A}$ has a suitable structure as an A - A -equivalence bimodule in this case. (See, for example, [27, Proposition 2.3].)

Let A be a C*-algebra, and let τ be a densely defined lower semicontinuous trace on A . Put $d_\tau(h) = \lim_{n \rightarrow \infty} \tau \otimes \text{Tr}(h^{\frac{1}{n}})$ for $h \in (A \otimes \mathbb{K})_+$. Then d_τ is a dimension function. The following proposition is a key proposition in this paper.

Proposition 2.1. Let A be a simple σ -unital C*-algebra with a unique tracial state τ and no unbounded trace. Define a map T of $\text{Pic}(A)$ to \mathbb{R}_+^\times by $T([\overline{h H_A}]) = d_\tau(h)$. Then T is a well-defined multiplicative map and $T([\mathcal{E}_{\text{id}}^A]) = 1$.

Proof. This is an immediate consequence of [27, Proposition 3.1 and Proposition 3.4] (put $[\mathcal{X}] = [\mathcal{E}_{\text{id}}^A]$). \square

Remark 2.2. [27, Proposition 3.1 and Proposition 3.4] are shown by using the countable bases of (right) Hilbert modules. See [14], [15] and [46] for bases of Hilbert modules.

Put $\mathcal{F}(A) = \text{Im}(T)$. Then we see that $\mathcal{F}(A)$ is equal to the set

$$\{d_\tau(h) \in \mathbb{R}_+^\times \mid h \text{ is a positive element in } A \otimes \mathbb{K} \text{ such that } A \cong \overline{h(A \otimes \mathbb{K})h}\}$$

by the results in [27]. We call $\mathcal{F}(A)$ the *fundamental group* of A , which is a multiplicative subgroup of \mathbb{R}_+^\times . We refer the reader to [29], [30] and [27] for details of the fundamental groups of C^* -algebras. If A is σ -unital, then $\mathcal{F}(A)$ is equal to the set of trace-scaling constants for automorphisms:

$$\mathfrak{S}(A) := \{\lambda \in \mathbb{R}_+^\times \mid \tau \otimes \text{Tr} \circ \alpha = \lambda \tau \otimes \text{Tr} \text{ for some } \alpha \in \text{Aut}(A \otimes \mathbb{K})\}.$$

3. THE CUNTZ SEMIGROUP

In this section we shall review basic facts of the Cuntz semigroups and some results in [6], [9], [36] and [38]. See, for example, [2] for details of the Cuntz semigroups. Let A be a C^* -algebra. For positive elements $a, b \in A$ we say that a is *Cuntz smaller than* b , written $a \precsim b$, if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of A such that $\|x_n^* b x_n - a\| \rightarrow 0$. Positive elements a and b are said to be *Cuntz equivalent*, written $a \sim b$, if $a \precsim b$ and $b \precsim a$. Define the *Cuntz semigroup* $\text{Cu}(A)$ as the set of Cuntz equivalence classes of positive elements in $A \otimes \mathbb{K}$ endowed with the order $[a] \leq [b]$ if a is Cuntz smaller than b , and the addition $[a] + [b] = [a' + b']$ where $a \sim a'$, $b \sim b'$ and $a'b' = 0$. Note that this definition is different from the original definition $W(A)$ in [7]. (We have $\text{Cu}(A) = W(A \otimes \mathbb{K})$.) The Cuntz semigroup $\text{Cu}(A)$ is also defined using Hilbert right A -modules (see [6]). For positive elements $a, b \in A \otimes \mathbb{K}$ we say that a is *compactly contained* in b , written $a \ll b$ if whenever $[b] \leq \sup_{n \in \mathbb{N}} [b_n]$ for an increasing sequence $\{[b_n]\}_{n \in \mathbb{N}}$, then there exists a natural number n such that $[a] \leq [b_n]$. Coward, Elliott and Ivanescu [6] showed that $\text{Cu}(A)$ has the following properties:

- (1) every increasing sequence in $\text{Cu}(A)$ has a supremum,
- (2) for any element $[a]$ in $\text{Cu}(A)$ there exists an increasing sequence $\{[a_n]\}_{n \in \mathbb{N}}$ of $\text{Cu}(A)$ such that $[a_n] \ll [a_{n+1}]$ for any $n \in \mathbb{N}$ and $[a] = \sup [a_n]$,
- (3) the operation of passing to the supremum of an increasing sequence and the relation \ll are compatible with addition.

Moreover they showed that $\text{Cu}(A)$ is a functor which is continuous with respect to inductive limits ([6, Theorem 2]). For a positive element $a \in A \otimes \mathbb{K}$ and $\epsilon > 0$ we denote by $(a - \epsilon)_+$ the element $f(a)$ in $A \otimes \mathbb{K}$ where $f(t) = \max\{0, t - \epsilon\}$, $t \in \sigma(a)$. Then we have $(a - \epsilon)_+ \ll a$.

Following the definition in [38], the Cuntz semigroup $\text{Cu}(A)$ is said to be *almost unperforated* if $(k + 1)[a] \leq k[b]$ for some $k \in \mathbb{N}$, then $[a] \leq [b]$. Rørdam showed that if A is \mathcal{Z} -stable, then $\text{Cu}(A)$ is almost unperforated (see [38, Theorem 4.5]). We denote by $T(A)$ the set of densely defined lower semicontinuous traces on A and $T_1(A)$ the set of tracial states on A . If A is simple exact C^* -algebra with traces, then $\text{Cu}(A)$ is almost unperforated if and only if A has strict comparison, that is, if $a, b \in (A \otimes \mathbb{K})_+$ with $d_\tau(a) < d_\tau(b)$ for any $\tau \in T(A)$, then $[a] \leq [b]$. (See [9, Proposition 4.2, Remark 4.3 and Proposition 6.2] and [38, Proposition 3.2 and Corollary 4.6].) The following proposition is an immediate corollary of [9, Theorem 6.6]. (Note that they considered the more general case.) But we shall give a self-contained proof based on their arguments (see also [9, Proposition 6.4]).

Proposition 3.1. Let A be a simple exact C*-algebra, and let a and b be positive elements in $A \otimes \mathbb{K}$. Assume that $\text{Cu}(A)$ is almost unperforated and 0 is an accumulation point of the spectrum $\sigma(a)$ of a . Then if $d_\tau(a) \leq d_\tau(b)$ for any $\tau \in T(A)$, then a is Cuntz smaller than b .

Proof. Let a and b be positive elements in $A \otimes \mathbb{K}$ such that $d_\tau(a) \leq d_\tau(b)$ for any $\tau \in T(A)$. We may assume that $\|a\| = \|b\| = 1$. For any $k \in \mathbb{N}$ we have

$$d_\tau(\text{diag}(\overbrace{a, \dots, a}^k)) = kd_\tau(a) \leq kd_\tau(b) < (k+1)d_\tau(b) = d_\tau(\text{diag}(\overbrace{b, \dots, b}^{k+1})).$$

Hence $k[a] \leq (k+1)[b]$ for any $k \in \mathbb{N}$ because A has strict comparison. Let $\epsilon > 0$, and choose a positive function c_ϵ on $\sigma(a)$ such that $c_\epsilon(t) > 0$ on $t \in (0, \epsilon)$ and $c_\epsilon(t) = 0$ on $\sigma(a) \setminus (0, \epsilon)$. Then we have $[c_\epsilon(a)] + [(a - \epsilon)_+] \leq [a]$. Note that for any $\epsilon > 0$, $c_\epsilon(a)$ is a nonzero positive element because 0 is an accumulation point of $\sigma(a)$. Hence we have $2[a] \leq \sup_{n \in \mathbb{N}} n[c_\epsilon]$ by the simplicity of A and Brown's theorem in [4]. There exists a natural number m such that $2[(a - \epsilon)_+] \leq m[c_\epsilon(a)]$ since $2[(a - \epsilon)_+] \ll 2[a]$. Therefore we have

$$(m+2)[(a - \epsilon)_+] \leq m[(a - \epsilon)_+] + m[c_\epsilon(a)] \leq m[a] \leq (m+1)[b].$$

By the assumption that $\text{Cu}(A)$ is almost unperforated, we see that $[(a - \epsilon)_+] \leq [b]$ for any $\epsilon > 0$, and hence we have $[a] \leq [b]$. \square

Corollary 3.2. Let A be a simple exact stably projectionless C*-algebra, and let a and b be positive elements in $A \otimes \mathbb{K}$. Assume that $\text{Cu}(A)$ is almost unperforated. Then if $d_\tau(a) = d_\tau(b)$ for any $\tau \in T(A)$, then a is Cuntz equivalent to b .

Proof. For any nonzero positive element a in $A \otimes \mathbb{K}$, 0 is an accumulation point of $\sigma(a)$ because A is a stably projectionless C*-algebra. Hence we obtain the conclusion by Proposition 3.1. \square

Based on the result in [36], we say that a C*-algebra A has *almost stable rank one* if for every σ -unital hereditary subalgebra $B \subseteq A \otimes \mathbb{K}$ we have $B \subseteq \overline{\text{GL}(\widetilde{B})}$. Robert showed that if A is a \mathcal{Z} -stable stably projectionless C*-algebra, then A has almost stable rank one (see [36, Corollary 4.5] and [38]). The following proposition is [36, Proposition 4.7]. See [6, Theorem 3] for the proof.

Proposition 3.3. Let A be a σ -unital C*-algebra such that A has almost stable rank one and a and b positive elements in $A \otimes \mathbb{K}$. Then a is Cuntz smaller than b if and only if there exists a right Hilbert A -module $\mathcal{X} \subseteq \overline{bH_A}$ such that \mathcal{X} is isomorphic to $\overline{aH_A}$ as a right Hilbert A -module, and a is Cuntz equivalent to b if and only if $\overline{aH_A}$ is isomorphic to $\overline{bH_A}$ as a right Hilbert A -module.

Corollary 3.2 and Proposition 3.3 are important in the proof of our main result. These propositions show that every separable simple \mathcal{Z} -stable stably projectionless C*-algebra A with a unique tracial state has similar properties of II_1 factors (Murray-von Neumann comparison theory). Moreover we have the following proposition.

Proposition 3.4. Let A be a simple exact separable stably projectionless C*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ . Assume that τ is normalized, $\text{Cu}(A)$ is almost unperforated, A has almost stable rank one and $\mathcal{F}(A) = \mathbb{R}_+^\times$. Then every nonzero hereditary subalgebra of A is isomorphic to A .

Proof. Corollary 3.2 and Proposition 3.3 imply if a and b are positive elements in $A \otimes \mathbb{K}$ such that $d_\tau(a) = d_\tau(b)$, then $\overline{aH_A}$ is isomorphic to $\overline{bH_A}$ as a right Hilbert A -module. Hence $K_A(\overline{aH_A})$ is isomorphic to $K_A(\overline{bH_A})$. Therefore if $d_\tau(a) = d_\tau(b)$,

then $\overline{a(A \otimes \mathbb{K})a}$ is isomorphic to $\overline{b(A \otimes \mathbb{K})b}$ because $K_A(\overline{aH_A})$ is isomorphic to $\overline{a(A \otimes \mathbb{K})a}$. Since $\mathcal{F}(A) = \mathbb{R}_+^\times$, for any $t \in (0, 1]$ there exists a positive element h such that $\overline{h(A \otimes \mathbb{K})h}$ is isomorphic to A and $d_\tau(h) = t$. Note that for any nonzero hereditary subalgebra B of a separable C^* -algebra A there exists a nonzero positive element h_0 in A such that B is isomorphic to $\overline{h_0 A h_0}$. Since $d_\tau(h_0) \in (0, 1]$, we obtain the conclusion. \square

4. MAIN RESULT

The following theorem is the main result in this paper. See [20, Corollary 4.8] and [29, Proposition 3.26] for the unital case.

Theorem 4.1. Let A be a simple exact σ -unital stably projectionless C^* -algebra with a unique tracial state τ and no unbounded trace. Assume that $\text{Cu}(A)$ is almost unperforated and A has almost stable rank one. Then the following sequence is exact:

$$1 \longrightarrow \text{Out}(A) \xrightarrow{\rho_A} \text{Pic}(A) \xrightarrow{T} \mathcal{F}(A) \longrightarrow 1.$$

Proof. It is clear that ρ_A is one-to-one, T is onto and $\text{Im}(\rho_A) \subseteq \text{Ker}(T)$. We shall show that $\text{Ker}(T) \subseteq \text{Im}(\rho_A)$. Let $[\mathcal{E}] \in \text{Ker}(T)$. Then Corollary 3.2 and Proposition 3.3 imply \mathcal{E} is isomorphic to $\overline{(h \otimes e_{11})H_A}$ as a right Hilbert A -module where h is a strict positive element in A and e_{11} is a rank one projection in \mathbb{K} because we have $d_\tau(h \otimes e_{11}) = 1$ by $\|\tau\| = 1$. It can easily be checked that $\overline{(h \otimes e_{11})H_A}$ is isomorphic to a right Hilbert A -module A with the obvious right A -action and $\langle a, b \rangle_A = a^*b$ for $a, b \in A$. Consequently we see that there exists some automorphism α such that $[\mathcal{E}] = [\mathcal{E}_\alpha^A]$, and hence $[\mathcal{E}] \in \text{Im}(\rho_A)$. \square

Corollary 4.2. Let A be a simple exact separable \mathcal{Z} -stable stably projectionless C^* -algebra with a unique tracial state τ and no unbounded trace. Then the following sequence is exact:

$$1 \longrightarrow \text{Out}(A) \xrightarrow{\rho_A} \text{Pic}(A) \xrightarrow{T} \mathcal{F}(A) \longrightarrow 1.$$

Proof. This is an immediate consequence of [38, Theorem 4.5], [36, Corollary 4.5] and Theorem 4.1. \square

Remark 4.3. There exists a unital simple AF algebra A with a unique tracial state such that $\text{Out}(A)$ is not a normal subgroup of $\text{Pic}(A)$. (See [28].) Of course A is a unital stably finite \mathcal{Z} -stable C^* -algebra. Therefore the corollary above shows that \mathcal{Z} -stable stably projectionless C^* -algebras are more well-behaved than unital stably finite \mathcal{Z} -stable C^* -algebras.

We shall show some examples.

Let \mathcal{W}_2 be the Razak-Jacelon algebra studied in [12], [35] and [36], which has trivial K -groups and a unique tracial state and no unbounded trace. The Razak-Jacelon algebra \mathcal{W}_2 is constructed as an inductive limit C^* -algebra of Razak's building block in [32], that is,

$$A(n, m) = \left\{ f \in C([0, 1]) \otimes M_m(\mathbb{C}) \mid f(0) = \text{diag}(\overbrace{c, \dots, c}^k, 0_n), f(1) = \text{diag}(\overbrace{c, \dots, c}^{k+1}), c \in M_n(\mathbb{C}) \right\}$$

where n and m are natural numbers with $n|m$ and $k := \frac{m}{n} - 1$. Let \mathcal{O}_2 denote the Cuntz algebra generated by 2 isometries S_1 and S_2 . There exists by universality a one-parameter automorphism group α of \mathcal{O}_2 given by $\alpha_t(S_j) = e^{it\lambda_j} S_j$. Kishimoto and Kumjian showed that if λ_1 and λ_2 are all nonzero of the same sign and λ_1 and λ_2 generate \mathbb{R} as a closed subgroup, then $\mathcal{O}_2 \rtimes_\alpha \mathbb{R}$ is a simple stable projectionless C^* -algebra with unique (up to scalar multiple) densely defined lower semicontinuous

trace in [18] and [19]. Moreover Robert [35] showed that $\mathcal{W}_2 \otimes \mathbb{K}$ is isomorphic to $\mathcal{O}_2 \rtimes_{\alpha} \mathbb{R}$ for some λ_1 and λ_2 . (See also [8].) In particular, $\mathcal{W}_2 \otimes \mathbb{K}$ has a one parameter trace scaling automorphism group σ (see [18]).

Theorem 4.4. The Picard group of Razak-Jacelon algebra \mathcal{W}_2 is isomorphic to a semidirect product of $\text{Out}(\mathcal{W}_2)$ with \mathbb{R}_+^{\times} . Moreover if A is a simple exact separable \mathcal{W}_2 -stable C*-algebra with a unique tracial state τ and no unbounded trace, then the Picard group of A is isomorphic to a semidirect product of $\text{Out}(A)$ with \mathbb{R}_+^{\times} .

Proof. Note that we see that A is stably projectionless C*-algebra because $A \otimes \mathbb{K} \cong A \otimes \mathcal{W}_2 \otimes \mathbb{K}$ has a one parameter trace scaling automorphism group $\text{id} \otimes \sigma$. Since \mathcal{W}_2 is \mathcal{Z} -stable, we have the following exact sequence:

$$1 \longrightarrow \text{Out}(A) \xrightarrow{\rho_A} \text{Pic}(A) \xrightarrow{T} \mathcal{F}(A) \longrightarrow 1$$

by Corollary 4.2. We have $\text{Pic}(A) \cong \text{Out}(A \otimes \mathbb{K})$ and $\mathcal{F}(A) = \mathfrak{S}(A)$ (see Section 2). Therefore we see that $\mathcal{F}(A) = \mathbb{R}_+^{\times}$ and the exact sequence above splits because $A \otimes \mathbb{K}$ has a one parameter trace scaling automorphism group. Consequently $\text{Pic}(A)$ is isomorphic to $\text{Out}(A) \rtimes \mathbb{R}_+^{\times}$. \square

Remark 4.5. (i) By the result of Brown, Rieffel and Green and the theorem above, we have

$$\text{Out}(\mathcal{W}_2 \otimes \mathbb{K}) \cong \text{Out}(\mathcal{W}_2) \rtimes \mathbb{R}_+^{\times}.$$

(ii) We do not assume that A is nuclear in the theorem above. Hence we have

$$\text{Pic}(\mathcal{W}_2 \otimes C_r^*(\mathbb{F}_n)) \cong \text{Out}(\mathcal{W}_2 \otimes C_r^*(\mathbb{F}_n)) \rtimes \mathbb{R}_+^{\times}$$

where \mathbb{F}_n is a non-amenable free group with n generators. Moreover Proposition 3.4 shows that every nonzero hereditary subalgebra of $\mathcal{W}_2 \otimes C_r^*(\mathbb{F}_n)$ is isomorphic to $\mathcal{W}_2 \otimes C_r^*(\mathbb{F}_n)$.

(iii) Let A be a simple unital AF algebra with two extremal tracial states. Then $\mathcal{W}_2 \otimes A$ is a simple stably projectionless C*-algebra with two extremal tracial states and in the class of Robert's classification theorem [35]. It can be checked that $\text{Out}(\mathcal{W}_2 \otimes A)$ is not a normal subgroup of $\text{Pic}(\mathcal{W}_2 \otimes A)$ by Robert's classification theorem and a similar proposition as [20, Proposition 1.5]. (We need to replace the K_0 -groups with the trace spaces.)

5. \mathcal{Z} -STABILITY OF STABLY PROJECTIONLESS C*-ALGEBRAS

In this section we shall generalize the result of Matui and Sato in [24] to stably projectionless C*-algebras. Note that our arguments are essentially based on their arguments.

We shall review some results of Kirchberg's central sequence algebra in [16]. For a separable C*-algebra A , set

$$c_0(A) := \{(a_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n \rightarrow \infty} \|a_n\| = 0\}, \quad A^\infty := \ell^\infty(\mathbb{N}, A)/c_0(A).$$

Let B be a C*-subalgebra of A . We identify A and B with the C*-subalgebras of A^∞ consisting of equivalence classes of constant sequences. Put

$$A_\infty := A^\infty \cap A', \quad \text{Ann}(B, A^\infty) := \{(a_n)_n \in A^\infty \cap B' \mid (a_n)_n b = 0 \text{ for any } b \in B\}.$$

Then $\text{Ann}(B, A^\infty)$ is an closed ideal of $A^\infty \cap B'$, and define

$$F(A) := A_\infty / \text{Ann}(A, A^\infty).$$

We call $F(A)$ the *central sequence algebra* of A . A sequence $(a_n)_n$ is said to be *central* if $\lim_{n \rightarrow \infty} \|a_n a - a a_n\| = 0$ for all $a \in A$. A central sequence is a representative of an element in A_∞ . Since A is separable, A has a countable approximate unit $\{h_n\}_{n \in \mathbb{N}}$. It is easy to see that $[(h_n)_n]$ is a unit in $F(A)$. If A is unital, then $F(A) = A_\infty$. Moreover we see that $F(A)$ is isomorphic to

$M(A)^\infty \cap A'/\text{Ann}(A, M(A)^\infty)$ since for any $(y_n)_n \in M(A)^\infty \cap A'$, $(y_n h_n)_n$ is a central sequence in A and $[(y_n)_n] = [(y_n h_n)_n]$ in $M(A)^\infty \cap A'/\text{Ann}(A, M(A)^\infty)$. Let $\{e_{ij}\}_{i,j \in \mathbb{N}}$ be the standard matrix units of \mathbb{K} . Define a map φ of $F(A)$ to $F(A \otimes \mathbb{K})$ by $\varphi([(x_n)_n]) = [(x_n \otimes \sum_{i=1}^n e_{ii})_n]$. Then it is easily seen that φ is a well-defined injective homomorphism. It can be checked that φ is surjective by using matrix units and the centrality of sequence because a similar argument as above shows any element in $F(A \otimes \mathbb{K})$ is equal to $[(\sum_{i,j=1}^n x_{n,i,j} \otimes e_{i,j})_n]$ for some sequence $\{x_{n,i,j}\}_{n \in \mathbb{N}}$ in A . Hence $F(A)$ is isomorphic to $F(A \otimes \mathbb{K})$. We shall show the following proposition (which is based on [45, Proposition 2.2]) by a similar way as in [37, Theorem 7.2.2]. See [16, Proposition 4.11] for more general cases.

Proposition 5.1. Let A be a separable C^* -algebra. If there exist a unital homomorphism of the prime dimension drop algebra $I(k, k+1)$ to $F(A)$ for any $k \in \mathbb{N}$, then A is \mathcal{Z} -stable.

Proof. By a similar argument as in [45, Proposition 2.2] and the construction of \mathcal{Z} in [13], we see that there exists a unital homomorphism α of \mathcal{Z} to $F(A)$.

Let φ be an injective homomorphism of A to $A \otimes \mathcal{Z}$ such that $\varphi(a) = a \otimes 1_{\mathcal{Z}}$, and put $C := M(A \otimes \mathcal{Z})^\infty \cap \varphi(A)'/\text{Ann}(\varphi(A), M(A \otimes \mathcal{Z})^\infty)$. Then we can regard α as a unital homomorphism of \mathcal{Z} to C since $F(A)$ is isomorphic to $M(A)^\infty \cap A'/\text{Ann}(A, M(A)^\infty)$. Define a unital homomorphism β of \mathcal{Z} to $M(A \otimes \mathcal{Z})^\infty \cap \varphi(A)'$ by $\beta(x) = (1_{M(A)} \otimes x)_n$, and let $[\beta] : \mathcal{Z} \rightarrow C$ be the quotient homomorphism of β . Then we see that $C^*(\alpha(\mathcal{Z}), [\beta](\mathcal{Z}))$ in C is isomorphic to $\mathcal{Z} \otimes \mathcal{Z}$. By the property of \mathcal{Z} , there exists a sequence $\{w_m\}_{m \in \mathbb{N}}$ of unitary elements in C such that $\lim_{m \rightarrow \infty} w_m^* [\beta](x) w_m = \alpha(x)$ for any $x \in \mathcal{Z}$ and w_m is in the connected component of 1_C in $U(C)$ for any $m \in \mathbb{N}$. Since w_m is in the connected component of 1_C in $U(C)$, there exists a unitary element u_m in $M(A \otimes \mathcal{Z})^\infty \cap \varphi(A)'$ such that $[u_m] = w_m$ for any $m \in \mathbb{N}$. For any $a \in A$, $x \in \mathcal{Z}$ and all $y \in M(A \otimes \mathcal{Z})^\infty \cap \varphi(A)'$ such that $[y] = \alpha(x)$, we have

$$y\varphi(a) = \lim_{m \rightarrow \infty} u_m^* \beta(x) u_m \varphi(a) = \lim_{m \rightarrow \infty} u_m^* \beta(x) \varphi(a) u_m = \lim_{m \rightarrow \infty} u_m^* (a \otimes x) u_m$$

by $[y] = \lim_{m \rightarrow \infty} [u_m^* \beta(x) u_m]$ and the definition of $\text{Ann}(\varphi(A), M(A \otimes \mathcal{Z})^\infty)$. Hence we see that $\lim_{m \rightarrow \infty} u_m^* (a \otimes x) u_m$ is an element in $\varphi(A)^\infty$. Therefore for any $z \in A \otimes \mathcal{Z}$, $\lim_{m \rightarrow \infty} d(u_m^* z u_m, \varphi(A)^\infty) = 0$. We obtain the conclusion by a similar argument as in [37, Proposition 2.3.5 and Proposition 7.2.1]. \square

If A is unital, every densely defined lower semicontinuous trace on A is bounded. Hence if A is simple and $A \otimes \mathbb{K}$ has a nonzero projection, then there exists a full hereditary subalgebra B of A such that every densely defined lower semicontinuous trace on B is bounded. In general, we have the following proposition.

Proposition 5.2. Let A be a σ -unital simple C^* -algebra. Then there exists a full hereditary subalgebra B of A such that every densely defined lower semicontinuous trace on B is bounded.

Proof. Let $\text{Ped}(A)$ be the Pedersen ideal of A , and let h be a nonzero positive element in $\text{Ped}(A)$. Then \overline{hAh} is contained in $\text{Ped}(A)$, and hence $\tau(b) < \infty$ for any $b \in \overline{hAh}_+$ and any $\tau \in T(A)$ because $\text{Ped}(A)$ is a minimal dense ideal. We refer the reader to [3] and [31] for properties of the Pedersen ideal. Define a map Φ of $T(A)$ to $T(\overline{hAh})$ by $\Phi(\tau) = \tau|_{\overline{hAh}}$. It can easily be checked that $\Phi(\tau)$ is equal to $\text{Tr}_\tau^\mathcal{X}$ constructed in [27, Proposition 2.4] where $\mathcal{X} = \overline{hAh}$. Therefore we see that Φ is a bijective map since A is simple (or h is full). (See also references in [27].) Consequently every densely defined lower semicontinuous trace on \overline{hAh} is bounded because every positive linear functional is automatically bounded. \square

If A is separable, then A is \mathcal{Z} -stable if and only if some full hereditary subalgebra is \mathcal{Z} -stable by Proposition 5.1 and Brown's theorem in [4] since $F(A)$ is isomorphic to $F(A \otimes \mathbb{K})$. (See also [44].) Therefore we may assume that A has no unbounded trace by the proposition above. Note that if A has strict comparison and no unbounded trace, then for any $a, b \in A_+$ satisfying $d_\tau(a) < d_\tau(b)$ for all $\tau \in T_1(A)$, we have $a \precsim b$.

Proposition 5.3. Let A be a separable C*-algebra such that $T_1(A)$ is a non-empty compact set, and let $\{h_m\}_{m \in \mathbb{N}}$ be a countable approximate unit for A and $\epsilon > 0$. Then there exists a natural number N such that

$$\max_{\tau \in T_1(A)} |\tau(f_n) - \tau(h_m f_n)| < \epsilon$$

for any $m \geq N$ and for any sequence $(f_n)_{n \in \mathbb{N}}$ of positive contractions in A . In particular, we have

$$\lim_{n \rightarrow \infty} \max_{\tau \in T_1(A)} |\tau(h_n f_n) - \tau(f_n)| = 0.$$

Proof. For any $\tau \in T_1(A)$, we have $\tau(h_m) \leq \tau(h_{m+1})$ and $\lim \tau(h_m) = 1$. By Dini's theorem, there exists a natural number N such that

$$\max_{\tau \in T_1(A)} |1 - \tau(h_m)| < \epsilon$$

for any $m \geq N$. For any sequence $(f_n)_{n \in \mathbb{N}}$ of positive contractions in A ,

$$\begin{aligned} \max_{\tau \in T_1(A)} |\tau(f_n) - \tau(h_m f_n)| &= \max_{\tau \in T_1(A)} |\tau((1 - h_m)^{1/2} f_n (1 - h_m)^{1/2})| \\ &\leq \max_{\tau \in T_1(A)} |1 - \tau(h_m)| < \epsilon. \end{aligned}$$

□

We denote by \tilde{A} the unitization algebra of A . Note that we consider $A = \tilde{A}$ when A is unital. We recall some definitions.

Definition 5.4. Let A be a separable C*-algebra with no unbounded trace. Assume that $T_1(A)$ is a non-empty compact set. We say that A has *property (SI)* if for any central sequences $(e_n)_n$ and $(f_n)_n$ of positive contractions in A satisfying

$$\lim_{n \rightarrow \infty} \max_{\tau \in T_1(A)} \tau(e_n) = 0, \quad \lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \min_{\tau \in T_1(A)} \tau(f_n^m) > 0,$$

there exists a central sequence $(s_n)_n$ in A such that

$$\lim_{n \rightarrow \infty} \|s_n^* s_n - e_n\| = 0, \quad \lim_{n \rightarrow \infty} \|f_n s_n - s_n\| = 0.$$

For a completely positive map φ of \tilde{A} to \tilde{A} , we say that φ *can be excised in small central sequences in A* if for any central sequences $(e_n)_n$ and $(f_n)_n$ of positive contractions in A satisfying the property above, there exists a sequence $(s_n)_{n \in \mathbb{N}}$ in A such that

$$\lim_{n \rightarrow \infty} \|s_n^* a s_n - \varphi(a) e_n\| = 0 \text{ for any } a \in \tilde{A}, \quad \lim_{n \rightarrow \infty} \|f_n s_n - s_n\| = 0.$$

Remark 5.5. In the definition above, it is important that e_n and f_n are elements in A . We see that if $\text{id}_{\tilde{A}}$ can be excised in small central sequences in A , then A has property (SI) (see [24, Proof of (iii) \Rightarrow (iv) of Theorem 1.1]).

We shall generalize [23, Lemma 4.6] and [24, Lemma 2.4] to non-unital C*-algebras.

Lemma 5.6. Let c be a positive element in a separable C^* -algebra A such that $T_1(A)$ is a non-empty compact set, and let $\theta \in \mathbb{R}$. For any central sequence $(f_n)_n$ of positive contractions in A , we have

$$\limsup_{n \rightarrow \infty} \max_{\tau \in T_1(A)} |\tau(cf_n) - \theta\tau(f_n)| \leq 2 \max_{\tau \in T_1(A)} |\tau(c) - \theta|.$$

Proof. Let $\{h_m\}_{m \in \mathbb{N}}$ be a countable approximate unit for A . A similar argument as in the proof of [23, Lemma 4.6] shows that

$$\limsup_{n \rightarrow \infty} \max_{\tau \in T_1(A)} |\tau(cf_n) - \theta\tau(h_m f_n)| \leq 2 \max_{\tau \in T_1(A)} |\tau(c) - \theta\tau(h_m)|$$

for any $m \in \mathbb{N}$. By Proposition 5.3, we have

$$\limsup_{n \rightarrow \infty} \max_{\tau \in T_1(A)} |\tau(cf_n) - \theta\tau(f_n)| \leq 2 \max_{\tau \in T_1(A)} |\tau(c) - \theta|.$$

□

Lemma 5.7. Let A be a separable simple C^* -algebra such that $T_1(A)$ is a non-empty compact set, and let a be a nonzero positive element in \tilde{A} . If $(f_n)_n$ is a central sequence of positive contractions in A such that

$$\lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \min_{\tau \in T_1(A)} \tau(f_n^m) > 0,$$

then

$$\lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \min_{\tau \in T_1(A)} \tau(f_n^{m/2} a f_n^{m/2}) > 0.$$

Proof. Put $R := a^{1/2}A$. Since A is simple, R is a right ideal of A such that $R^*R = AaA$ is a dense ideal of A . Therefore there exists a sequence $\{v_j\}_{j \in \mathbb{N}}$ in A such that $\{\sum_{j=1}^n v_j^* a v_j\}_{n \in \mathbb{N}}$ is an approximate unit for A by a similar argument as in [4, Lemma 2.3]. By Proposition 5.3, there exists a natural number N such that

$$\lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \min_{\tau \in T_1(A)} \tau\left(\sum_{j=1}^N v_j^* a v_j f_n^m\right) > 0.$$

We have

$$\begin{aligned} \lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \min_{\tau} \tau\left(\sum_{j=1}^N v_j^* a v_j f_n^m\right) &= \lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \min_{\tau} \sum_{j=1}^N \tau(v_j^* a^{1/2} f_n^m a^{1/2} v_j) \\ &= \lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \min_{\tau} \sum_{j=1}^N \tau(f_n^{m/2} a^{1/2} v_j v_j^* a^{1/2} f_n^{m/2}) \\ &\leq \sum_{j=1}^N \|v_j\|^2 \lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \min_{\tau} \tau(f_n^{m/2} a f_n^{m/2}). \end{aligned}$$

Hence we obtain the conclusion. □

Let A be a separable simple C^* -algebra, and let τ be a tracial state on A . Consider the GNS representation $(\pi_\tau, H_\tau, \xi_\tau)$ associated with τ . Then $\pi_\tau(A)''$ is a finite von Neumann algebra and $\pi_\tau(A)$ is strongly dense subalgebra of $\pi_\tau(A)''$ in general. Indeed, every approximate unit for $\pi_\tau(A)$ is strongly convergent to 1_{H_τ} . We can identify $C^*(\pi_\tau(A), 1_{H_\tau})$ in $B(H_\tau)$ with its unitization algebra \tilde{A} . Therefore we obtain the following lemma (which is based on Haagerup's theorem ([11, Theorem 3.1])) by the proof of [40, Lemma 2.1]. See also [25, Proposition 3.5 and Theorem 4.3].

Lemma 5.8. ([40, Lemma 2.1])

Let A be a separable simple nuclear C*-algebra, and let τ be a tracial state on A . For any sequence $\{H_n\}_{n \in \mathbb{N}}$ of positive contractions in $\pi_\tau(A)''$ such that $\|[H_n, x]\|_\tau \rightarrow 0$ for all $x \in \pi_\tau(A)''$, there exists a central sequence $(c_n)_n$ of positive contractions in A such that $\|c_n - H_n\|_\tau \rightarrow 0$.

Note that we need to consider the unitization algebra \tilde{A} in order to use [10, Theorem 2.1] in the proof of the lemma above. But we see that $(c_n)_n$ is contained in A by the construction of $(c_n)_n$ in the proof of [40, Lemma 2.1].

If τ is an extremal tracial state on a separable simple infinite-dimensional nuclear C*-algebra A , then $\pi_\tau(A)''$ is the AFD Π_1 factor in general. Therefore Lemma 5.8 and essentially the same proof as [24, Lemma 3.3] show the following lemma. See also the proof of [25, Proposition 4.8].

Lemma 5.9. ([24, Lemma 3.3])

Let A be a separable simple infinite-dimensional nuclear C*-algebra with finitely many extremal tracial states. For any $k \in \mathbb{N}$, there exist central sequences $(c_{i,n})_n$ in A , $i = 1, 2, \dots, k$ such that $c_{1,n}$ is a positive contraction for any $n \in \mathbb{N}$, $(c_{i,n}c_{j,n}^*)_n = \delta_{i,j}(c_{1,n}^2)_n$ and

$$\lim_{n \rightarrow \infty} \max_{\tau \in T_1(A)} |\tau(c_{1,n}^m) - \frac{1}{k}| = 0$$

for any $m \in \mathbb{N}$.

Let ω be a pure state on A . Then we can uniquely extend ω to a pure state $\tilde{\omega}$ on \tilde{A} . Moreover if A is a separable simple non-type I C*-algebra, then $\pi_\omega(A) \cap K(H_\omega) = \{0\}$. Therefore the same proof as [24, Lemma 3.1] shows that every completely positive map of \tilde{A} to \tilde{A} can be approximated in the pointwise norm topology by completely positive map φ of the form

$$\varphi(x) = \sum_{l=1}^N \sum_{i,j=1}^N \tilde{\omega}(d_i^* x d_j) c_{l,i}^* c_{l,j}, \quad x \in \tilde{A}$$

where $c_{l,i}, d_i \in \tilde{A}$. Therefore we see that a similar argument as in [24] implies the following theorem by using Lemma 5.6, Lemma 5.7 and Lemma 5.9 instead of [23, Lemma 4.6], [24, Lemma 2.4] and [24, Lemma 3.3].

Theorem 5.10. Let A be a separable simple infinite-dimensional nuclear C*-algebra with finitely many extremal tracial states and no unbounded trace. If A has a strict comparison, then any completely positive map of \tilde{A} to \tilde{A} can be excised in small central sequences in A .

The following theorem is the main theorem in this section.

Theorem 5.11. Let A be a separable simple infinite-dimensional non-type I nuclear C*-algebra with a finite dimensional lattice of densely defined lower semicontinuous traces. Then A has strict comparison if and only if A is \mathcal{Z} -stable.

Proof. Rørdam showed that if A is \mathcal{Z} -stable, then A has strict comparison (see [38, Corollary 4.6]). We shall show the only if part. By Proposition 5.2, we may assume that A has no unbounded trace. Hence A has property (SI) by Remark 5.5 and Theorem 5.10. For any $k \in \mathbb{N}$, there exist central sequences $(c_{i,n})_n$ in A , $i = 1, 2, \dots, k$ such that $c_{1,n}$ is a positive contraction, $(c_{i,n}c_{j,n}^*)_n = \delta_{i,j}(c_{1,n}^2)_n$ and

$$\lim_{n \rightarrow \infty} \max_{\tau \in T_1(A)} |\tau(c_{1,n}^m) - \frac{1}{k}| = 0$$

for any $m \in \mathbb{N}$ by Lemma 5.9. Let $\{h_n\}_{n \in \mathbb{N}}$ be an approximate unit for A . Taking a suitable subsequence of $\{h_n\}_{n \in \mathbb{N}}$, we may assume that $(h_n)_n(c_{1,n})_n =$

$(c_{1,n})_n(h_n)_n$. Define central sequences $(f_{i,n})_n$ in A , $i = 1, \dots, k$ by $(f_{i,n})_n := (c_{i,n}h_n^{1/2})_n$, and put $(e_n)_n := (h_n - \sum_{i=1}^k f_{i,n}^* f_{i,n})_n$. Then we may assume that $(e_n)_n$ is a central sequence of positive contractions in A . Proposition 5.3 implies $\lim_{n \rightarrow \infty} \max_{\tau} |\tau(f_{i,n}^* f_{i,n} - c_{i,n}^* c_{i,n})| = 0$ for any $1 \leq i \leq k$, and hence we have

$$\lim_{n \rightarrow \infty} \max_{\tau \in T_1(A)} \tau(e_n) = 0.$$

Note that $(f_{1,n})_n$ is a central sequence of positive contractions in A by the assumption of $(h_n)_n$. Because $\{h_n^{1/2}\}_{n \in \mathbb{N}}$ is also an approximate unit for A , we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \max_{\tau} \|c_{1,n} - c_{1,n}^{1/2} h_n^{1/2} c_{1,n}^{1/2}\|_{\tau}^2 &= \limsup_{n \rightarrow \infty} \max_{\tau} \tau((c_{1,n} - c_{1,n}^{1/2} h_n^{1/2} c_{1,n}^{1/2})^2) \\ &\leq \limsup_{n \rightarrow \infty} \max_{\tau} \tau(c_{1,n} - c_{1,n}^{1/2} h_n^{1/2} c_{1,n}^{1/2}) \\ &= 0 \end{aligned}$$

by Proposition 5.3. Hence

$$\begin{aligned} \limsup_{n \rightarrow \infty} \max_{\tau} |\tau(c_{1,n}^m) - \tau(f_{1,n}^m)| &= \limsup_{n \rightarrow \infty} \max_{\tau} |\tau(c_{1,n}^m - (c_{1,n} h_n^{1/2})^m)| \\ &= \limsup_{n \rightarrow \infty} \max_{\tau} |\tau(c_{1,n}^m - (c_{1,n}^{1/2} h_n^{1/2} c_{1,n}^{1/2})^m)| \\ &\leq \limsup_{n \rightarrow \infty} \max_{\tau} \|c_{1,n}^m - (c_{1,n}^{1/2} h_n^{1/2} c_{1,n}^{1/2})^m\|_{\tau} = 0 \end{aligned}$$

for any $m \in \mathbb{N}$. Therefore we have

$$\lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \min_{\tau \in T_1(A)} \tau(f_{1,n}^m) = 1/k > 0.$$

Since A has property (SI), there exists a central sequence $(s_n)_n$ in A such that $(s_n^* s_n + \sum_{i=1}^k f_{i,n}^* f_{i,n})_n = (h_n)_n$ and $(f_{1,n} s_n)_n = (s_n)_n$. We have $[(f_{i,n} f_{j,n}^*)_n] = \delta_{i,j} [(f_{1,n}^2)_n]$ and $[(s_n^* s_n + \sum_{i=1}^k f_{i,n}^* f_{i,n})_n] = 1$ in $F(A)$ because $[(h_n^{1/2})_n]$ is a unit in $F(A)$. It follows from [39, Proposition 2.1] that there exists a unital homomorphism of $I(k, k+1)$ to $F(A)$. Consequently A is \mathcal{Z} -stable by Proposition 5.1. \square

Remark 5.12. Let A be a separable simple infinite-dimensional non-type I nuclear C^* -algebra with a finite dimensional lattice of densely defined lower semicontinuous traces, that has strict comparison. Since A is \mathcal{Z} -stable by the theorem above, there exists a unital homomorphism of \mathcal{Z} to $M(A)^{\infty} \cap A'$. But we do not know that we could show this fact directly without using Kirchberg's central sequence algebras. Note that if A is non-unital, then there exists no unital homomorphism of \mathcal{Z} to $(\tilde{A})^{\infty} \cap A'$ because \tilde{A} is not \mathcal{Z} -stable.

The following corollary is an immediate consequence of the theorem above and Corollary 4.2.

Corollary 5.13. Let A be a separable simple nuclear stably projectionless C^* -algebra with a unique tracial state and no unbounded trace. Assume that A has strict comparison. Then we have the following exact sequence:

$$1 \longrightarrow \text{Out}(A) \xrightarrow{\rho_A} \text{Pic}(A) \xrightarrow{T} \mathcal{F}(A) \longrightarrow 1.$$

We shall consider some examples. We say that A is a *1-dimensional NCCW complex* if A is a pullback C^* -algebra of the form

$$\begin{array}{ccc} A & \xrightarrow{\pi_2} & E \\ \downarrow \pi_1 & & \downarrow \rho \\ C([0, 1]) \otimes F & \xrightarrow{\delta_0 \oplus \delta_1} & F \oplus F \end{array}$$

where E and F are finite-dimensional C*-algebras and δ_i is the evaluation map at i . Razak's building block $A(n, m)$ is a 1-dimensional NCCW complex. Robert classified inductive limits of 1-dimensional NCCW complexes with trivial K_1 -groups in [35].

Corollary 5.14. Let A be a simple C*-algebra with a finite dimensional lattice of densely defined lower semicontinuous traces, that is expressible as an inductive limit C*-algebra of 1-dimensional NCCW-complexes with trivial K_1 -groups. Then A is \mathcal{Z} -stable.

Proof. The Cuntz semigroup of a 1-dimensional NCCW-complex is almost unperforated (see, for example, [1]). Hence we see that $\text{Cu}(A)$ is almost unperforated by the continuity of the functor Cu with respect to inductive limits ([6, Theorem 2]) and an argument based on compact containments. Therefore A has strict comparison, and hence we obtain the conclusion by Theorem 5.11. \square

Corollary 5.15. Let A be a simple stably projectionless C*-algebra with a unique tracial state and no unbounded trace, that is expressible as an inductive limit C*-algebra of 1-dimensional NCCW-complexes with trivial K_1 -groups and B a separable simple C*-algebra with a unique tracial state and no unbounded trace. Then we have the following exact sequence:

$$1 \longrightarrow \text{Out}(A \otimes B) \xrightarrow{\rho_{A \otimes B}} \text{Pic}(A \otimes B) \xrightarrow{T} \mathbb{R}_+^\times \longrightarrow 1.$$

Proof. Robert's classification theorem ([35, Corollary 6.2.4]) shows that every non-zero hereditary subalgebra of A is isomorphic to A . Therefore $\mathcal{F}(A) = \mathbb{R}_+^\times$ because it is clear that A has a positive element with a continuous spectrum. Since $\mathcal{F}(A \otimes B) = \mathbb{R}_+^\times$ and $A \otimes B$ is separable, $A \otimes B$ is a stably projectionless C*-algebra by [27, Corollary 4.10]. Therefore we obtain the conclusion by Corollary 4.2 and Corollary 5.14. \square

We do not know whether the exact sequence above splits. This question is related to the existence of a one parameter trace scaling automorphism group of $A \otimes \mathbb{K}$. For any countable abelian groups G_1 and G_2 , Kishimoto showed that there exists a stable projectionless simple separable nuclear C*-algebra A with unique (up to scalar multiple) densely defined lower semicontinuous trace with $K_0(A) = G_1$ and $K_1(A) = G_2$ in [17]. These stably projectionless C*-algebras are constructed as the crossed products $\mathcal{O} \rtimes_\alpha \mathbb{R}$ by certain one parameter automorphism groups α of Kirchberg algebras \mathcal{O} and the dual actions of α are trace scaling actions of $\mathcal{O} \rtimes_\alpha \mathbb{R}$. Hence it is natural to believe that there exists a kind of duality between \mathcal{Z} -stable stably projectionless C*-algebras (with unique trace) and \mathcal{O}_∞ -stable C*-algebras. From this view point, it seems to be possible to introduce the stably projectionless C*-algebra \mathcal{W}_n for any $n \geq 3$. Hence we denote by \mathcal{W}_2 the Razak-Jacelon algebra. On the other hand, Tikuisis [43] constructed a simple separable nuclear stably projectionless C*-algebra whose Cuntz semigroup is not almost unperforated.

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